

Knot Complement Problem for L-space $\mathbb{Z}HS^3$

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In this paper we look at the knot complement problem for L-space \mathbb{Z} -homology spheres. We show that an L-space \mathbb{Z} -homology sphere Y cannot be obtained as a non-trivial surgery along a knot $K \subset Y$. As a consequence, we prove that knots in an L-space \mathbb{Z} -homology sphere are determined by their complements.

1 Introduction

A knot K in a 3-manifold Y is *determined by its complement* if the existence of a homeomorphism between $Y \setminus K$ and $Y \setminus K'$ for some other knot K' , implies the existence of homeomorphism between the pair (Y, K) and (Y, K') . Here we do not expect these homeomorphisms to be orientation preserving. The “*knot complement problem*” is the problem of knowing if a given knot is determined by its complement. Note that in this context two knots are equivalent if there is an homeomorphism of the ambient manifold which send one to the other and the homeomorphism need not be orientation preserving. Gordon and Luecke [4] have proved that non-trivial knots in S^3 and $S^2 \times S^1$ are determined by their complements. D. Matignon has proved in [6] that, if one only considers orientation preserving homeomorphisms, then non-trivial non-hyperbolic knots are determined by their complements in closed, atoroidal and irreducible Seifert fibred 3-manifolds; except the axes in $L(p, q)$ when $q^2 \equiv \pm 1 \pmod{p}$. In general there are some knots which are not determined by their complements. In [16] Y. Rong classified all such Seifert fibered knots in closed 3-manifolds other than lens spaces. In [5] D. Matignon give examples of hyperbolic knots in lens spaces which are not determined by their complements. Here we are interested in a problem related to the knot complement problem for L-space \mathbb{Z} -homology spheres. A example of this are knots in the Poincaré sphere $\Sigma(2, 3, 5)$. More precisely, we prove the following theorems and corollaries.

Theorem 1.1 *Let K be a knot in an oriented L-space \mathbb{Z} -homology sphere Y and let $r \in \mathbb{Q}$. The result of an r -surgery along K is never homeomorphic to Y .*

From this theorem we can answer the oriented knot complement problem for L-space \mathbb{Z} -homology spheres.

Theorem 1.2 *Knots in L-space \mathbb{Z} -homology spheres are determined by their oriented complements.*

In particular this is true for the Poincaré sphere $\Sigma(2,3,5)$ which is the only known irreducible L-space \mathbb{Z} -homology sphere apart from S^3 . Note also that $\Sigma(2,3,5)$ does not admit orientation reversing homeomorphisms.

Corollary 1.3 *Let K be a knot in $\Sigma(2,3,5)$ and let $r \in \mathbb{Q}$. The result of an r -surgery along K is never homeomorphic to $\Sigma(2,3,5)$.*

Corollary 1.4 *Knots in $\Sigma(2,3,5)$ are determined by their complements.*

In the earlier version of the paper Theorem 1.1 and Theorem 1.2 was stated only for orientation preserving homeomorphisms. Fyodor Gainullin [3] independently proved that more generally null-homologous knots in a rational homology L-space are determined by their oriented complements. Note that Lens spaces are rational homology L-spaces but the axes in Lens spaces are not null-homologous and they are knot determined by their complements by [6]. We prove here the special case of L-space \mathbb{Z} -homology spheres. The main part of the proof here, Lemma 3.3, was a lemma in the author Ph.D. thesis and is a direct consequence of a previous work of J. Rasmussen. The main theorem is obtained by using properties of genus 1 fibred knots together with Lemma 3.3.

Organization

The paper is organized as follows. In section 2 we give some preliminaries. In section 3 we give the proof of the main theorems.

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2 Preliminaries

Heegaard Floer homologies.

Heegaard Floer homology is an invariant for closed oriented three manifolds Y . The invariant, denoted \widehat{HF} is the homology of a chain complex defined from an Heegaard splitting of Y but also admits some combinatorial definition. Ozsváth-Szabó in [8] and Rasmussen in [14] defined a related invariant for null-homologous knots K in Y , taking the form of an induced filtration on the Heegaard Floer complex of Y . The filtered chain homotopy type of this complex is a knot invariant, the “knot Floer homology”.

Knot Floer homology then associates to a null-homologous knot K in Y a $\mathbb{Z} \oplus \mathbb{Z}$ -filtered complex $CFK^\infty(Y, K)$, generated over \mathbb{Z} by elements $[\mathbf{x}; i, j]$ where $x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ and $(i, j) \in \mathbb{Z} \oplus \mathbb{Z}$. The set $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$ is the intersection of two half dimensional totally real submanifolds in a symplectic manifold obtained from an Heegaard splitting of Y . Let \mathfrak{G} be the set of generator of $CFK^\infty(Y, K)$, the $\mathbb{Z} \oplus \mathbb{Z}$ -filtration on the complex is given by the map $\mathcal{F} : \mathfrak{G} \longrightarrow \mathbb{Z} \oplus \mathbb{Z}$ defined by $\mathcal{F}([\mathbf{x}; i, j]) = (i, j)$. The complex $\widehat{CFK}(Y, K)$ is a subcomplex of a quotient complex of $CFK^\infty(Y, K)$ and is equipped with a \mathbb{Z} -filtration. The homology of $\widehat{CFK}(Y, K)$ is denoted $\widehat{HFK}(Y, K)$ and its Euler characteristic is the normalized Alexander polynomial $\Delta_K(T)$ of the knot K with the \mathbb{Z} -filtration corresponding to the exponent of T . From more details on the subject we refer to [[10], [9], [8], [12], [13], [14]].

***L*-space.**

The Heegaard Floer homology $\widehat{HF}(Y)$ has the property that

$$\text{rank} \widehat{HF}(Y) \geq |H_1(Y; \mathbb{Z})|.$$

The manifolds which satisfy the equality form an important class of manifold in 3-manifold theory.

Definition 2.1 *A rational homology sphere Y is called an L -space if*

$$\text{rank} \widehat{HF}(Y) = |H_1(Y; \mathbb{Z})|.$$

Being an L -space is also equivalent to $\widehat{HF}(Y, \mathfrak{s}) \cong \mathbb{Z}$ for each Spin^c structure \mathfrak{s} on Y .

L -spaces are Heegaard Floer analogues of lens-spaces. In particular every Lens space $L(p, q)$ with $p \neq 0$ is an L -space. They also include double branched cover of non-split alternating links.

A knot $K \subset Y$ is said to admit an L -space surgery if for some rational number r , $Y_K(r)$ is an L -space. We will use the fact that such knot has a very special knot Floer homology.

Irreducibility of $Y \setminus K$

Before continuing further let us discuss the irreducibility of $Y \setminus K$. The knot complement $Y \setminus K$ is irreducible if and only if it does not lie in a ball. Indeed if $Y \setminus K$ was reducible then there are two 3-manifolds M and N such that M is distinct from S^3 , $K \subset N$ and $Y = M \sharp (N \setminus K)$. However meridional surgery on K yields Y again so $Y_K(\infty) = Y$. Thus

$$Y_K(\infty) = Y = M \sharp N_K(\infty).$$

But Y is irreducible and M is distinct from S^3 . Therefore $N_K(\infty) \cong S^3$ and it follows that $N \cong S^3$. Therefore K lies in a ball. Conversely if a non-trivial knot K in Y lies in ball, then its the complement is obviously reducible.

The problem for a non-trivial knot which lies in a ball is then equivalent to the problem for knots in S^3 . Since it is known [4] that no non-trivial surgery on S^3 can reproduce S^3 , we can assume that our knot K does not lie in a ball. That is we can assume $Y \setminus K$ is irreducible.

3 Proof of main theorems

We need the following characterization of $\widehat{HFK}(Y, K)$ for a K is a knot in an integral homology L -space Y . It was proved in [[11] theorem 1.2] for the case of S^3 and stated in [[17] proposition 3.7] for the more general case of L -space homology sphere.

Proposition 3.1 *Let K be a knot in a L -space \mathbb{Z} -homology sphere Y . If $Y_K(r)$ is an L -space for some rational number r , then there is an increasing sequence of integers $n_{-k} < \dots < n_k$ such that $n_i = -n_{-i}$, and $\widehat{HFK}(K, j) = 0$ unless $j = n_i$ for some i , in which case $\widehat{HFK}(K, j) \cong \mathbb{Z}$.*

An immediate corollary [[17] corollary 3.8] is a simplified expression for the Alexander polynomials of such knots.

Corollary 3.2 *Let K be a knot in a L -space \mathbb{Z} -homology sphere. If K admits an L -space surgery, then the Alexander polynomial of K has the form*

$$\Delta_K(T) = (-1)^k + \sum_{j=1}^k (-1)^{k-j} (T^{n_j} + T^{-n_j}),$$

for some increasing sequence of positive integers $0 < n_1 < n_2 < \dots < n_k$.

Our key lemma is the following which is about a characterisation of the Alexander polynomial of knots in L -space \mathbb{Z} -homology spheres which admit L -space \mathbb{Z} -homology sphere surgery.

Lemma 3.3 *Let Z be an L -space \mathbb{Z} -homology sphere and let K be any non-trivial knot in Z . If K admits an L -space \mathbb{Z} -homology sphere non-trivial surgery, then $\Delta_K(T) = T^{-1} - 1 + T$, K has genus 1 and $\widehat{HFK}(K) \cong \mathbb{Z}^3$.*

Here the Alexander polynomial is normalized so that $\Delta_K(1) = 1$ and $\Delta_K(T) = \Delta_K(T^{-1})$.

Proof Rasmussen proved in [[15] Proposition 4.5] that:

If Z is an L -space with $H_1(Z) = \mathbb{Z}/p\mathbb{Z}$, and $K \subset Z$ is a *primitive knot* (i.e K generates $H_1(Z)$) with a homology sphere non-trivial surgery X . Then X is an L -space if and only if one of the following condition holds:

- (1) $\widehat{HFK}(K) \cong \mathbb{Z}^p$ and $\text{width } \widehat{HFK}(K) < 2p$.
- (2) $\widehat{HFK}(K) \cong \mathbb{Z}^{p+2}$, $\text{width } \widehat{HFK}(K) = 2p$.

Here $\text{width } \widehat{HFK}(K)$ is the difference $\text{Max} - \text{Min}$, where Max is the maximum value of j for which $\widehat{HFK}(K, j)$ is nontrivial and Min is the minimum value.

On the other hand we also have a formula for the genus of K , $g(K) = (\text{width } \widehat{HFK}(K) - p + 1)/2$. See [15] theorem 4.3.

In our case $p = 1$ so all the hypothesis are satisfied and we have either

- (1) $\widehat{HFK}(K) \cong \mathbb{Z}$ and $\text{width } \widehat{HFK}(K) < 2$.
- (2) $\widehat{HFK}(K) \cong \mathbb{Z}^3$, $\text{width } \widehat{HFK}(K) = 2$.

The first case implies that $\text{width } \widehat{HFK}(K) = 0$ then $g(K) = 0$. Therefore we are left with the second case, we can then compute the Euler characteristic of $\widehat{HFK}(K)$ to obtain the symmetrized Alexander polynomial of K using the formula

$$\Delta_K(T) = \sum_{i,j} (-1)^i \dim \widehat{HFK}_i(K, j) T^j,$$

we obtain

$$\Delta_K(T) = a_{j_0} T^{j_0} + a_{j_0+1} T^{j_0+1} + a_{j_0+2} T^{j_0+2},$$

for some j_0 , using the fact that $\Delta_K(T) = \Delta_K(T^{-1})$ we get $j_0 = -1$. Since $\text{width } \widehat{HFK}(K) = 2$ we have $a_{-1} = a_1 = \pm 1$. so

$$\Delta_K(T) = \pm T^{-1} + a_0 \pm T.$$

On the other hand From corollary 3.2

$$\Delta_K(T) = (-1)^k + \sum_{j=1}^k (-1)^{k-j} (T^{n_j} + T^{-n_j}),$$

for some increasing sequence of positive integers $0 < n_1 < n_2 < \dots < n_k$. Therefore $j = 1, k = 1, n_j = 1$ and

$$\Delta_K(T) = T^{-1} - 1 + T.$$

Now computing the second derivative gives $\Delta_K''(1) = 2$. Finally since $\text{width } \widehat{HFK}(K) = 2$ and $g(K) = \max\{k \mid \widehat{HFK}_*(K, k) \neq 0\}$, we must have $g(K) = 1$. \square

Theorem 1.1 *Let K be a knot in an oriented L -space \mathbb{Z} -homology sphere Y and let $r \in \mathbb{Q}$. The result of an r -surgery along K is never homeomorphic to Y .*

Proof Since Y is an L-space \mathbb{Z} -homology sphere, lemma 3.3 implies that K is a genus one knot, $\Delta_K(T) = T^{-1} - 1 + T$ and $\widehat{HFK}(K) \cong \mathbb{Z}^3$. Since we can assume $Y \setminus K$ is irreducible, the result of Yi Ni about fibred knot [[7] Theorem 1.1] applies. Therefore K is a genus one fibred knot since $\widehat{HFK}(K) \cong \mathbb{Z}^3$. By chapter 5 of [2], since $\Delta_K(T) = T^{-1} - 1 + T$, the monodromy of the fibered knot K is the monodromy of the trefoil knot so Y_K is the trefoil exterior. It then follows that $Y_K(r)$ is cannot be homeomorphic to Y .

□

From this theorem we can answer the oriented knot complement problem for knots in L-space \mathbb{Z} -homology spheres.

Theorem 1.2 *Knots in L-space \mathbb{Z} -homology spheres are determined by their oriented complements.*

Proof Let K and K' be two non-trivial knots in an L-space \mathbb{Z} -homology sphere Y , let us denote V and V' their complements with the induced orientations. Suppose there is an orientation preserving homeomorphism $f : V \rightarrow V'$. Let μ_K , respectively $\mu_{K'}$, be the meridional slope of K , respectively K' , and let $r = f(\mu_K)$. The oriented manifold $V'(r)$ is orientation preserving homeomorphic to Y and therefore by Theorem 1.1 $r = \pm\mu_{K'}$. It follows that we can extend f to an orientation preserving homeomorphism between (Y, K) and (Y, K') . □

For the special case of $\Sigma(2, 3, 5)$ Kenneth Baker ([1] Lemma 13) showed that there is a unique genus one fibred knot in $\Sigma(2, 3, 5)$ which is the surgery dual to the negative trefoil. This gives another proof that knots in $\Sigma(2, 3, 5)$ are determined by their complements.

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